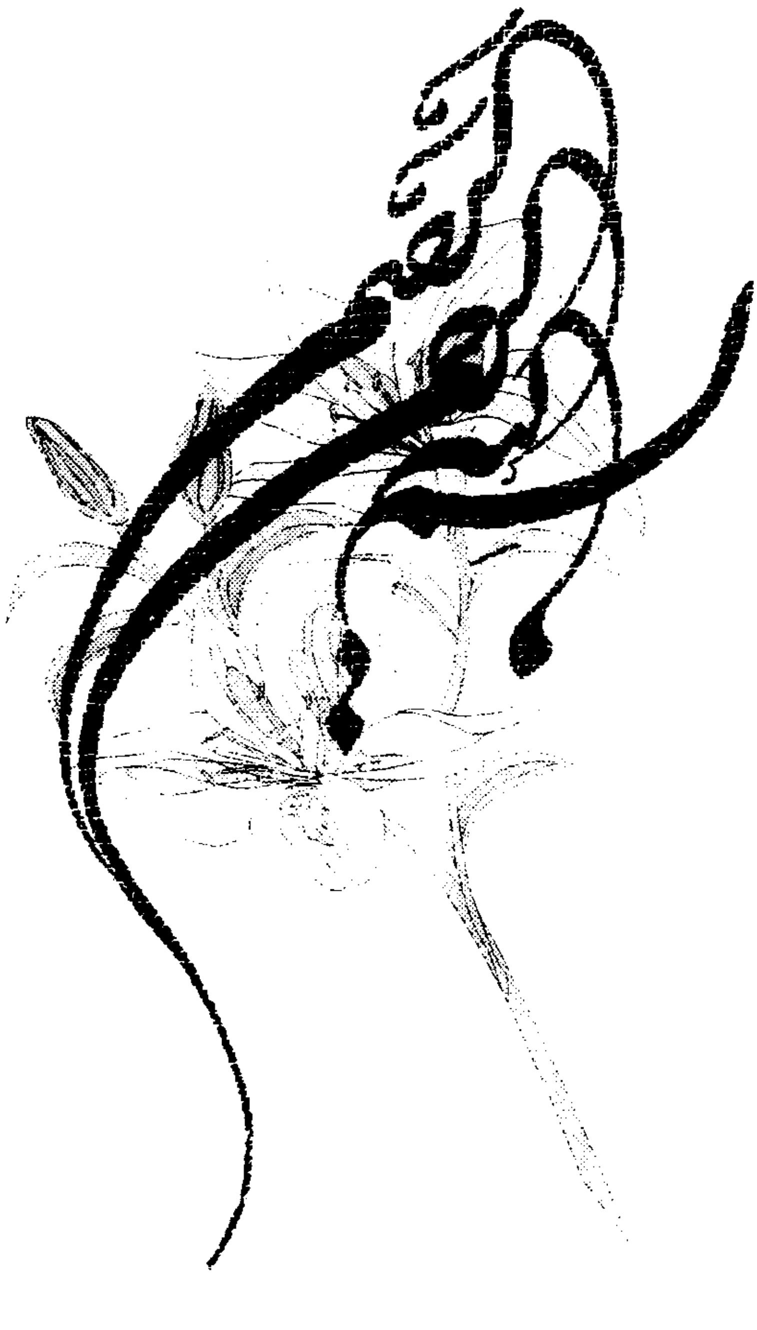

In the Name of Allah

Ferdowsi University of Mashhad

Faculty of
Mathematical sciences

Extensions, Minimality and Idempotents of Certain Semigroup Compactifications

Mohammad Reza Miri

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Directed by :
Professor M.A. Pourabddolah

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To the memory of my father

PREFACE

Harmonic analysis is primarily the study of function and measures on the topological–algebraic structures. Here, structure is given by a semitopological semigroup, i.e. a semigroup with a topology rendering continuous left and right translations.

The general theme on which the thesis based is some properties of semigroup compactifications. Each chapter consists of some sections and starts with a very short introduction, which describes the material included. We have tried to make the text self-contained, as far as possible, however we have broken this principle in a few places which have not been essential for later developments. For this, in chapter one we introduce some background material which is needed in the later chapters. For notation and terminology we shall follow Berglund et al. [5], as far as possible. However there are some exceptions, for example see (1.5.4). Nonetheless readers who are familiar with [5], may start reading from chapter two, without loss of continuity. We know that if S is a subsemigroup of a semitopological semigroup T, and $\mathcal{F}$ stands for one of the spaces $\mathcal{AP}$, $\mathcal{WAP}$, $\mathcal{SAP}$, $\mathcal{D}$ or $\mathcal{LC}$, and $(\epsilon, T^{\mathcal{F}})$ denotes the canonical $\mathcal{F}$-compactification of T, where T has the property that $\mathcal{F}(S) = \mathcal{F}(T)_{|S}$, then $(\epsilon_{|S}, \overline{\epsilon(S)})$ is an $\mathcal{F}$-compactification of S [5; chapter 5]. In chapter two we try to show the converse of this problem when T is a locally compact group and S is a closed normal subgroup of T. In this way we construct various semigroup compactifications of T from the same type compactifications of S. This enables us to obtain specific information about the structure of the compactifications of G from the structure of the compactifications of N. A noncompact locally compact group G is minimally weakly almost periodic if each weakly almost periodic function can be written as $g + h$ where g is almost periodic and h vanishes at infinity. In the first section of chapter two we will see that a noncompact locally compact solvable group is minimally weakly almost periodic if and only if $G/K(G) \cong M(2)$, where $K(G)$ is the largest compact subgroup of G, a result proved by Chou in [8]. Also in second and third sections of this chapter we generalize the notion of minimally (weakly) almost periodic group to the corresponding notion for semigroups.

The results of this section are mainly adaptations of those of Prof. H.D. Junghenn, in a paper titled " Direct sume of spaces of Functions on semigroups" in Semigroup Forum Vol. **49**(1994) 115-123, the reference to which was forgotten until after the completion of the printing process.

Since weakly almost periodic functions are hard to discover, some of the most significant advances have come through indirect methods. This was the case with results about idempotents in $\mathbb{N}^{\mathcal{WAP}}$. (The operation in the commutative semigroup $\mathbb{N}^{\mathcal{WAP}}$ is written +, so that an idempotent e satisfies e=e+e.) The question of whether $\mathbb{N}^{\mathcal{WAP}}$ could contain more than one idempotent was raised by West, [51], in connection with a problem in operator theory about the existence of projections, and he found a positive solution in that context. Techniques from harmonic analysis allied with the operator theoretic approach enabled Brown and Moran, in a series of papers, to say much about the lattice structure of the semigroup of idempotents and in particular to show that this semigroup was infinite (see [6]).

However, the difficulty of these metods held out little hope of further progress, and indeed no deep new facts about idempotents were discoverd for nearly 20 years. Then Ruppert made the important breakthrough: he found a class of weakly almost periodic functions which could be described in elementay terms [49]. His direct approach not only allowed him to refine the results of Brown and Moran but gave new information on the relationship between idempotents and group topologies on $\mathbb{Z}$. Their functions were defined by infinite products taking values in [0, 1], and the proof that their functions were weakly almost periodic was hard. In chapter four we try to present a theory which is on the same general lines as Ruppert's but which is in some ways simpler. There are two main differences. First the base for the expansion of the integer is taken to be the negative number (-2). This allows the digits to remain positive, and once the simple rules for carrying out addition to a negative base have been worked out, the necessary calculations become easier. Secondly the interval [0, 1] is replaced by $[0, \infty]$. It has the advantage that infinite sums can be considered instead of infinite products. The weakly almost periodic functions which arise have a very simple form (Theorem 4.1.8).

Ruppert did not solve all the basic problems about idempotents in $\mathbb{N}^{\mathcal{WAP}}$ in [49]. For example, he left open the question of whether the set of idempotents is closed. This appears in the list of problems about semitopological semigroups compiled by Berglund in 1980 [3], and was raised again by Ruppert in 1984 monograph [45] and his survey article [47]. That is the question which this section answers negatively, for both $\mathbb{N}^{\mathcal{WAP}}$ and $\mathbb{Z}^{\mathcal{WAP}}$.

Throughout this chapter we shall work with $\mathbb{Z}$ rather than $\mathbb{N}$, as Ruppert did in [49], because $\mathbb{Z}$ arises naturally when expansions to negative bases are considered. The conclusion we desire for $\mathbb{N}$ is an easy consequence of the result for $\mathbb{Z}$ (Corollary 4.1.11).

Finally, I take this occasion to record my debt to many people who have asissted me during the period of my research. First and foremost, my supervisor Professor M.A. Pourabdollah who deserves my profound gratitude for his enormous help, valuable guidance, and encouragement. Many mathematiticion who have contributed to me; I mention in particular, Professors J.F. Bergland, H.D.

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Contents

CHAPTER ONE

PRELIMINARIES

In this chapter we introduce some background material which is needed in the later chapters. We hope that this will save the reader unnecessary refererences to various textbooks. For notation and terminology we shall follow Berglund et al. [5], as far as possible.

1. Semitopological semigroup

By a semigroup we shall mean a non-empty set S with an associative binary operation, usually refferd to as the multiplication of S, defined on it. Let S be a semigroup, for $s, t \in S$ we write $\rho_t(s) = st = \lambda_s(t)$. For subsets A, B of S we write $As = \rho_s(A) = \{ts : t \in A\}$ and $sA = \lambda_s(A) = \{st : t \in A\}$, and $AB = \{st : s \in A, t \in B\}$. The *center* of S which is denoted by $Z(S)$, is defined by $Z(S)$={s$\in S : st = ts$ for each $t \in S\}$. S is called *commutative*, if $Z(S) = S$. An element $e \in S$ is said to be an *idempotent* if $e^2 = e$, the set of all idempotents of S is denoted by $E(S)$. An element $1 \in S$ is called left (resp. right) *identity* of S if $1s = s$ (resp. $s1 = s$) for all $s \in S$. A left identity that is also a right identity is called an identity.

S is called left (resp. right) *simple* if $Ss = S$ (resp. $sS = S$) for all $s \in S$. S is called a left (resp. right) *group* if for each pair $s, t \in S$ there exists a unique $x \in S$ such that $xs = t$ (resp. $sx = t$). It is easy to check that, a semigroup S is a left group if and only if it is left simple and contains an idemptent [5; 1.2.19] and a semigroup is a group if and only if it is both left simple and right simple.

Let S, T be two semigroups, a map $\theta : S \longrightarrow T$ is called a *homomorphism* if $\theta(st) = \theta(s)\theta(t)$ for all $s, t \in S$. A homomorphism that is one-to-one and onto is said to be an *isomorphism*. A homomorphism from S into the unit circle of $\mathbb{C}$ is called a *character* of S.

Let σ be a homomorphism of the semigroup T into the semigroup of all homomorphisms from S into itself; we write σ_s instead of $\sigma(s)$ for $s \in T$. Then $S \times T$ with multiplication $(s,t)(s_1,t_1) = (s\sigma_t(s_1), tt_1)$ is a semigroup which is called a *semidirect product* of S and T, and is denote by $S \times_\sigma T$. Note that $S \times_\sigma T$ reduces to the direct product if σ_t is the identity mapping on S, for all $t \in T$, it is obvious that if S and T have identities (each denoted by 1), then $(1,1)$ is an identity for $S \times_\sigma T$ if and only if σ_1 is the identity on S and $\sigma_t(1) = 1$ for all $t \in T$. As in the direct product case we shall denote by $P_1 : S \times_\sigma T \longrightarrow S$ and $P_2 : S \times_\sigma T \longrightarrow T$ the projection mappings, and by $q_1 : S \longrightarrow S \times_\sigma T$ and $q_2 : T \longrightarrow S \times_\sigma T$ the injection mappings, $q_1(s) = (s, 1), q_2(t) = (1, t)$. Note that, q_1, q_2 and P_2 are homomorphisms, but in general P_1 is not; and that it distinguishes the semidirect product from the direct product.

Now let S be a semigroup with a Hausdorff topology, then S is said to be right (resp. left) topological if the mapping ρ_t (resp. λ_t) is continuous; semitopological if both λ_s and ρ_s are continuous, and topological if the multiplication is jointly continuous. Right, left and semitopological groups can be defined similar to the corresponding notion for semigroup by exchanging semigroups with groups. A topological group is a group which is a topological semigroup, and the mapping $s \mapsto s^{-1} : S \longrightarrow S$ is also continuous.

1.1. Proposition (Ellis)

Every locally compact, Hausdorff, semitopological group, is a topological group,[13].

1.2. Example

$S_\infty = S \cup \{\infty\}$, *the one point compactification of a locally compact Hausdorff right topological semigroup* S, *is a semigroup which is a topological semigroup if* S *is a group and if* S *has a right zero, then* S_∞ *is not a topological semigroup,*[5; 1.3.3(d)].

Note that a semidirect product of two semitopological semigroups, need not be a semitopological semigroup, [35; example 8], it is easy to see that if either S is a topological semigroup and $(s,t) \longrightarrow \sigma_t(s) : S \times T \longrightarrow S$ is separately continuous, or T is discrete and σ_t is continuous for each $t \in T$, then the semidirect product of S and T is a semitopological semigroup.

Let S and T be two right topological semigroup and let $\theta : S \longrightarrow T$ be a homeomorphism and an isomorphism, then θ is called a topological isomorphism. If such a mapping exists, then we say that S and T are topologically isomorphic.

The next lemma collects some well-known results from [5]

1.3. Surjective lemma

Let S and let T be semigroups and $\theta : S \longrightarrow T$ be a surjective homomorphism.

(i) Let each of S and T have a minimal left ideal and a minimal right ideal.

(a) If L is a minimal left ideal of S, then $\theta(L)$ is a minimal left ideal of T.

(b) If R is a minimal right ideal of S, then $\theta(R)$ is a minimal right ideal of T.

(c) If I is a minimal ideal of S, then $\theta(I)$ is a minimal ideal of T.

(d) If G is a maximal group in a minimal ideal of S, then $\theta(G)$ is a maximal group in the corresponding minimal ideal of T.

Furthermore, the mappings $L \longrightarrow \theta(L)$, etc., in (a)-(d) of ideals in T and of groups in S to groups in T are surjective.

(ii) Let S and T have topologies with T being compact, let θ be continuous, and suppose that there is a compact $K \subset S$ with $\theta(K) = T$.

(a) If S is right topological, T is right topological.

(b) If S is left topological, T is left topological.

(c) If S is semitopological, T is semitopological.

(d) If S is topological, T is topological.

(e) If $s \longrightarrow sx$ is continuous in S, $t \longrightarrow t\theta(x)$ is continuou[illegible] T.

(f) If $s \longrightarrow xs$ is continuous in S, $t \longrightarrow \theta(x)t$ is continuous in T.

For our need in the algebraic theory of semigroups (resp. groups), we shall follow Howie [23] (resp. Robinson,[44]).

2. Flows

Let S be a semigroup and let X be a topological space, by an action of S on X, we mean a mapping $\pi : S \times X \longrightarrow X$, such that $\pi(s,.) : X \longrightarrow X$ is continuous for all $s \in S$, and $\pi(st,x) = \pi(s,\pi(t,x))$ for all $s,t \in S$ and $x \in X$. A *Flow* is a triple (S,X,π), where π is an action of S on the compact, Hausdorff space X. If X is also a convex subset of a locally convex topological vector space such that each mapping $\pi(s,.)$ is affine, then the flow (S, X, π) is said to be an *affine flow*. We shall usually denote a flow (S, X, π) by (S, X), if the action π being understood.

The *enveloping semigroup* $\Sigma(S, X)$ of the flow (S, X) is defined as the closure of $\{\pi(s,.) : s \in S\}$ in the product space X^X. Therefore, $\Sigma(S, X)$is always a compact right topological subsemigroup of X^X (under the coposition of maps), and $s \mapsto \pi(s,.) : S \longrightarrow \Sigma(S, X)$ is a homomorphismon onto a dense subsemigroup of $\Sigma(S, X)$ contained in $\Lambda(\Sigma(S, X))$. Furtheremore, the closure of $\pi(S, x)$ is equal to $\Sigma(S, X)(x)$, for all $x \in X$,[5; 1.6.5].

2.1. Distal flows

A flow (S, X), is called *distal* if for a pair $x, y \in X$, the existence of a net $\{s_\alpha\}$ in S such that $lim_\alpha s_\alpha x = lim_\alpha s_\alpha y$ implies $x = y$.

2.2. Proposition

A flow is distal if and only if its enveloping semigroup is a group, see,[5; 1.6.9] or [10; 5.3].

3. Admissibility of function spaces

Let $\mathcal{F}$ be a subspace of $B(S)$ (the $C^\star$- algebra of all bounded complex valued functions on the semigroup S), for every $s \in S$, left and right translation operators L_s and R_s on $B(S)$ are defined by $L_s f(t) = f(st)$ and $R_s f(t) = f(ts), (t \in S, f \in B(S))$. $\mathcal{F}$ is called *left (resp. right) translation* *invariant* if for all $s \in S$, $L_s\mathcal{F} \subset \mathcal{F}$ (resp. $R_s\mathcal{F} \subset \mathcal{F}$). If $\mathcal{F}$ is both left and right translation invariant, it is called *translation invariant*.

Let $\mathcal{F}$ be translation invariant. For $\mu \in \mathcal{F}^*$, the left and right *introversion operators* T_μ and U_μ, from $\mathcal{F}$ into $B(S)$ are defined by $(T_\mu f)(s) = \mu(L_s f)$ and $(U_\mu f)(s) = \mu(R_s f)$, $(s \in S, f \in \mathcal{F})$.

3.1. Proposition

Let $\mathcal{F}$ be a translation invariant subalgebra (resp. subspace) of $B(S)$, then for each $f \in \mathcal{F}$, $\{T_\mu f : \mu \in S^{\mathcal{F}}\}$ (resp. $\{T_\mu f : \mu \in aS^{\mathcal{F}}\}$) is the pointwise closure in $B(S)$ of $R_S f$ (resp. $co(R_S f)$), see [5; 2.2.3].

3.2. Remark

Let S be a semitopological semig. .p, and for each $f \in C(S)$, let X_f denote the pointwise closure of $R_S f$ in $C(S)$, then (Proposition 3.1) implies that X_f is compact (with the relative pointwise topology); it is easy to see that (S, X_f) is a flow under the natural action $(s,x) \mapsto R_s x : S \times X_f \longrightarrow X_f$.

A translation invariant subspace $\mathcal{F}$ is said to be left (resp. right) *introverted* if $T_\mu\mathcal{F} \subset \mathcal{F}$ (resp. $U_\mu\mathcal{F} \subset \mathcal{F}$) for all $\mu \in \mathcal{F}^*$ (or $\mu \in M(\mathcal{F})$ (the set of all mean on $\mathcal{F}$ which is denoted by $aS^{\mathcal{F}}$ in [5])). If $\mathcal{F}$ is also an algebra, and the inclusion holds for $\mu \in MM(\mathcal{F})$ (the set of all multiplicative mean on $\mathcal{F}$ denoted by $S^{\mathcal{F}}$ in [5]), $\mathcal{F}$ is called introverted (resp. m-introverted).

An *admissible subspace* of $B(S)$ is a conjugate closed, translation invari-