IN THE NAME OF GOD

PARTIAL ACTIONS OF GROUPS AND ACTIONS OF INVERSE SEMIGROUPS

BY:

YOUSEF YADOLLAHI

THESIS

SUBMITTED TO THE OFFICE OF VICE-CHANCELLOR FOR GRADUATE STUDIES IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE *DEGREE OF MASTER OF SCIENCE (M.S.)*

IN

PURE MATHEMATICS
SHIRAZ UNIVERSITY
SHIRAZ, IRAN

EVALUATED AND APPROVED BY THE THESIS COMMITTEE AS:EXCELLENT

B. Tabatabaie .... TABATABAIE, B., Ph.D., ASSISSTANT PROF. OF MATH. (CHAIRMAN)

.... TAGHAVI, M., Ph.D., ASSOCIATE PROF. OF MATH.

....B. Khani............ KHANI-ROBATI, B. Ph.D., ASSOCIATE PROF. OF MATH.

B. Yousefi............ YOUSEFI, B., Ph.D., ASSOCIATE PROF. OF MATH.

JANUARY 2003

ACKNOWLEDGMENT

First, I am profoundly indebted to Dr. Tabatabaie, my respectable supervisor who patiently and several versions of my thesis and guided me with constructive comments.

I am also grateful to Dr. Khani and Dr. Yousefi and Dr. Taghavi for the invaluable comments they made on the pre-final draft of my thesis.

My special thanks go to Miss. Nikmanesh who typed my thesis and Mr. Abdali who duplicated this thesis.

Finally I would like to express my debt to my mother, father, sister and my wife for their kind co-operation and encouragement.

To:

My Dear Mother

ABSTRACT

PARTIAL ACTIONS OF GROUPS AND ACTIONS OF INVERSE SEMIGROUPS

BY:

YOUSEF YADOLLAHI

One of our major goals in this thesis is to introduce for each group G an inverse semigroup associated to G which we will denote it by $S(G)$. We show that if G is a finite group of order p, then $S(G)$ has $2^{p-2}(p+1)$ elements. The concept of partial actions of groups on C^*-algebras are generalized to concept of partial actions of groups on sets.

Then we show that for every group G and any set X, there is a one-to-one correspondence between partial actions of G on X and actions of $S(G)$ on X.

A_e is constructed as an auxiliary C^*-algebra, and the partial group C^*-algebra of G is the C^*-algebra $C_p^*(G)$ given by the crossed product of A_e by θ, that is,

$$C_p^*(G) = A_e \times_\theta G.$$

In this thesis we show that there is a one-to-one correspondence between partial representations of G on H and representations of $S(G)$ on H and

C^*-algebra representations of $C_p^*(G)$ on H.

While the usual group C^*-algebra of finite commutative groups forgets everything but the order of the group, we show that the partial group C^*-algebra of the two commutative groups of order four, namely $\frac{z}{4z}$ and $\frac{z}{2z} \oplus \frac{z}{2z}$, are not isomorphic.

TABLE OF CONTENTS

CHAPTER I

FUNDAMENTAL CONCEPTS

1. FUNDAMENTAL CONCEPTS

In this chapter we study the basic concepts which we need in later chapters.

We begin by definitions of semigroups and inverse semigroups in section 1. In section 2 we define the groups defined via generators and relations. We study some properties of C^*-algebras and we define Hilbert bimodules in section 3. Section 4 is devoted to the study of concepts of partial actions of groups and investigate some properties of partial C^*-crossed products.

In section 5 we briefly study, the C^*-crossed products.

1.1. Inverse Semigroups

Definition 1.1.1. A semigroup is a nonempty set S together with a binary operation on S which is associative, that is, $a(bc) = (ab)c$ for all a, b, c in S.

Definition 1.1.2. If ϕ is a mapping from a semigroup $(S, 0)$ into a semigroup $(T, *)$ we say that ϕ is a homomorphism if

$$\phi(x \circ y) = \phi(x) * \phi(y)$$

for all x, y in S.

If ϕ is one-one we shall call it a monomorphism, and if it is both one-one and onto we shall call it an isomorphism.

Definition 1.1.3. If $(S,0)$ is a semigroup, then a nonempty subset T of S is called a subsemigroup of S if it is closed with respect to multiplication i.e. if $x \circ y \in T$, for all x, y in T.

Definition 1.1.4. If A is an arbitrary non-empty subset of a semigroup S, then the family of subsemigroups of S containing A is nonemtpy, since S itself is one such semigroup, hence the intersection of the family is a subsemigroup of S containing A. We denote it by $< A >$.

The subsemigroup $< A >$ consists of all elements of S that can be expressed as finite products of elements in A. If $< A >= S$ we shall say that A is a set of generators for S or a generating set of S.

Definition 1.1.5. An element a of semigroup S is called regular if there exists x in S such that $axa = a$. The semigroup S is called regular if all its elements are regular.

Definition 1.1.6. A subset A of a semigroup S is called

(i) right unitary if sa is in A then s is in A for all a in A and s in S.

(ii) left unitary if as is in A then s is in A for all a in A and s in S.

(iii) unitary if it is both left and right unitary.

An element e in a semigroup is said to be idempotent if $e^2 = e$.

Definition 1.1.7. A regular semigroup S is said to be E-unitary if the set E of idempotents is a unitary subsemigroup of S.

Definition 1.1.8. A Clifford semigroup is a regular semigroup S in which the idempotents are central, i.e., in which $ex = xe$ for every idempotent e and every element x in S.

Definition 1.1.9. A semigroup S is said to be an inverse semigroup provided there exists, for each x in S, a unique element x^* in S such that

(i) $xx^*x = x$

(ii) $x^*xx^* = x^*$.

Proposition 1.1.10. Let S be an inverse semigroup and E be the set of idempotents of S, then

(a) $(s^*)^* = s$ for every s in S.

(b) $e^* = e$ for every e in E.

Proof. See Proposition 1.4 of [11, Chapter V, Section 1].

Proposition 1.1.11. If $s_1, s_2, \ldots, s_n$ are elements of an inverse semigroup S, then

$$(s_1 s_2 \ldots s_n)^* = s_n^* s_{n-1}^* \ldots s_2^* s_1^*.$$

Proof. See Corollary 1.5 of [11, Chapter V, Seciton 1].

Proposition 1.1.12. Let S be an inverse semigroup, let T be a semigroup and let $\phi : S \longrightarrow T$ be a homomorphism. Then $\phi(S)$ is an inverse semigroup.

Proof. See Proposition 1.6 of [11, Chapter V, Section 1].

1.2. The Group Defined Via Generators and Relations

In this section, we briefly introduce, the concept of the group defined via generators and relations. For more information about groups defined via generators and relations may see [16].

Let $a, b, c, \ldots$ be distinct symbols and form the new symbols $a^{-1}, b^{-1}, c^{-1}, \ldots$. A word W in the symbols $a, b, c, \ldots$ is a finite sequence

$$f_1, f_2, \ldots, f_{n-1}, f_n \tag{1}$$

where each of the f_v is one of the symbols

$$a, b, c, \ldots, a^{-1}, b^{-1}, c^{-1}, \ldots;$$

the length $L(W)$ of W is the integer n. For convenience we introduce the empty word of length zero and denote it by 1.

If we wish to exhibit the symbols involved in W we write $W(a, b, c, \ldots)$. It is customery to write the sequence (1) without the commas as

$$f_1 f_2 \ldots f_{n-1} f_n. \tag{2}$$

It is also customry to abbreviate a block of n consecutive symbols a by a^n, and to abbreviate a block of n consecutive symbols a^{-1} by a^{-n}; e.g., the word $a^3 b^2 b^{-1} a^{-2} c^{-1}$ is the same as the word $aaabbb^{-1}a^{-1}a^{-1}c^{-1}$, but is different from the word $a^3 ba^{-2} c^{-1}$.

Thus aa^{-1} is a word in a of length two; and 1 is a word in a and b and c of length zero.

The inverse W^{-1} of a word W given by (2) is the word

$$f_n^{-1} f_{n-1}^{-1} \ldots f_2^{-1} f_1^{-1} \tag{3}$$

where if f_v is a or a^{-1} then f_v^{-1} is a^{-1} or a, respectively, and similarly if f_v is one of the symbols b or b^{-1}, c or $c^{-1}, \ldots$; the inverse of the empty word is itself.

For example $(aa^{-1}b)^{-1} = b^{-1}aa^{-1}$, $1^{-1} = 1$.

Clearly, $L(W) = L(W^{-1})$, and $(W^{-1})^{-1} = W$.

If W is the word $f_1 f_2 \ldots f_n$ and U is the word $f'_1 f'_2 \ldots f'_r$ then we define their juxtaposed product WU as the word $f_1 f_2 \ldots f_n f'_1 f'_2 \ldots f'_r$.

Clearly, $(WU)^{-1} = U^{-1}W^{-1}$ and $L(WU) = L(W) + L(U)$.

Given a mapping α of the symbols $a, b, c, \ldots$ into a group G with

$$\alpha(a) = g, \alpha(b) = h, \alpha(c) = k, \ldots$$

then we say that (under α) a defines g, b defines h, c defines $k, \ldots$, a^{-1} defines g^{-1}, b^{-1} defines h^{-1}, c^{-1} defines $k^{-1}, \ldots$; moreover, if W is given by (2) then W defines the element, denoted $W(g, h, k, \ldots)$, in G given by

$$g_1 g_2 \cdots g_{n-1} g_n,$$

where f_v defines g_v; the empty word 1 defines the identity element 1 of G.

Clearly, if the words U and V define the elements p and q of G, then U^{-1} defines p^{-1} and UV defines pq.

If every element of G is defined by some word in $a, b, c, \ldots$, then $a, b, c, \ldots$ are called generating symbols for G (under α) and $g, h, k, \ldots$ are called generating elements for G; if the context makes it clear, both generating symbols and generating elements may be referred to as generators for G.

For example, if V is the Klein four-group of elements $x, -x, \frac{1}{x}, \frac{-1}{x}$, that x is the identity element of V, then under the mapping $a \longrightarrow -x$, $b \longrightarrow \frac{1}{x}$, the word a^2 defines the element x and the word ab defines the element $-\frac{1}{x}$; therefore a and b are generating symbols for V (under this mapping).

A word $R(a, b, c, \ldots)$ which defines the identity element 1 in G is called a relator.

The equation

$$R(a, b, c, \ldots) = S(a, b, c, \ldots)$$

is called a relation if the word RS^{-1} is a relator (or equivalently, if R and S define the same element in G).

We can show that if given an arbitrary set of symbols, and an arbitrary prescribed set of words in these symbols, there is a unique group (up to isomorphism) with the symbols as generators and the set of prescribed words as defining relators.

1.3. C^*-Algebras and Hilbert Bimodules

In this section we define C^*-algebra. We begin by definition of Banach algebra.

Definition 1.3.1. A Banach algebra is an algebra A over $\mathbb{F}$ that has a norm $\|\cdot\|$ relative to which A is a Banach space and such that:

$$\|ab\| \leq \|a\|\|b\|,$$

for all a, b in A.

If a Banach algebra A has an identity e, i.e., $ae = ea = a$ for all a in A, then it is assumed that $\|e\| = 1$.

Definition 1.3.2. By an involution on an algebra A, we mean a mapping

$x \longrightarrow x^*$, from A into A such that:

i) $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$;

ii) $(xy)^* = y^* x^*$;

iii) $(x^*)^* = x$,

whenever x, y are in A, α is in $\mathbb{C}$ and $\bar{\alpha}$ denotes the complex conjugation of α.

Definition 1.3.3. A C^*-algebra is a complex Banach $*$-algebra A that satisfies the condition

$$\|a^* a\| = \|a\|^2,$$

for $a \in A$.

The above condition is called C^*-condition.

Example 1.3.4. The simplest example of a C^*-algebra is $\mathbb{C}$. In this algebra we have $\alpha^* = \bar{\alpha}$ and $\|\alpha\| = |\alpha|$ for all α in $\mathbb{C}$.

Example 1.3.5. The set of all n-tuples with complex coordinate, $\mathbb{C}^n$, is a C^*-algebra with the following structure:

$$\|(c_1, c_2, \ldots, c_n)\| = \max\{|c_i| : i = 1, 2, \ldots, n\},$$

$$(c_1, c_2, \ldots, c_n)(c_1', c_2', \ldots, c_n') = (c_1 c_1', c_2 c_2', \ldots, c_n c_n'),$$

$$(c_1, c_2, \ldots, c_n)^* = (\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n).$$

Example 1.3.6. If X is a compact space. The set $C(X)$ of all continuous complex-valued functions on X is a C^*-algebra with the following structure:

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x);$$